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A note on the steady high-Reynolds-number flow about a circular cylinder

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The most detailed theoretical description of the steady flow at high Reynolds number about a circular cylinder is given by Smith (1979). It appears to give a satisfactory description of much of the flow field. Numerical solutions for this flow computed by Fornberg (1985) are in conflict with some aspects of Smith's model.

An acknowledged weak point in Smith's argument is the lack of a detailed flow for the interior of the eddies, particularly with respect to their rear closure. This paper discusses this aspect of the flow with particular emphasis on the vorticity distribution and with the aid of a solution of the Navier–Stokes equations which is valid at the rear of the eddies. As a result an interpretation of Fornberg's results is possible. The self-induced velocity of translation of the vorticity distribution is considered and discussed with reference to the flow as $Re \rightarrow \infty$.

1. Introduction

The steady flow at high Reynolds number about a circular cylinder is an academic problem. Real high-Reynolds-number flows are unsteady. On the other hand, a full understanding of this academic problem should provide valuable insight into other real flows. Smith (1979) summarizes the history of the subject and gives the most detailed model of the flow field. Particular attention is paid to the separation region on the cylinder's surface, but a full discussion is given of the rest of the flow field since the effect of the eddies is to make a significant correction to the value of the free stream in the vicinity of the cylinder. To a first approximation Smith describes the eddies as a slender ellipse of stagnant fluid. A weak point in the argument is the absence of a model for the flow at the rear of the eddy. An exact solution of the Navier–Stokes equations which seems to be relevant to that region is given here.

Fornberg (1980, 1985) presents numerical solutions for the flow about a circular cylinder for Reynolds numbers up to 150 (Fornberg 1980) and 300 (Fornberg 1985).† Up to $Re = 160$ there is reasonable agreement with Smith's model; in particular the eddy length L and width W vary like $a\,Re$ and $a\,Re^{\frac{1}{2}}$ respectively, see figure 1. However, as is clear in figure 1 the computational results for $Re > 160$ disagree with Smith's hypothesis. The eddy width begins to increase rapidly with Reynolds number.

This note discusses the vorticity in the flow field on an order-of-magnitude basis. The eddy closure, which eluded Smith, is described in terms of an exact solution of the Navier–Stokes equation. The discussion is based on vorticity and continuity. Near the cylinder the Kirchhoff free-streamline model, as extended by Smith, is used as a starting point. However, the discussion leads to a descriptive model of the eddy

† We follow Smith and use $Re = aU/\nu$, where U is the flow velocity, a the diameter of the cylinder and ν the kinematic viscosity; many other authors and Fornberg use $R = 2aU/\nu$.

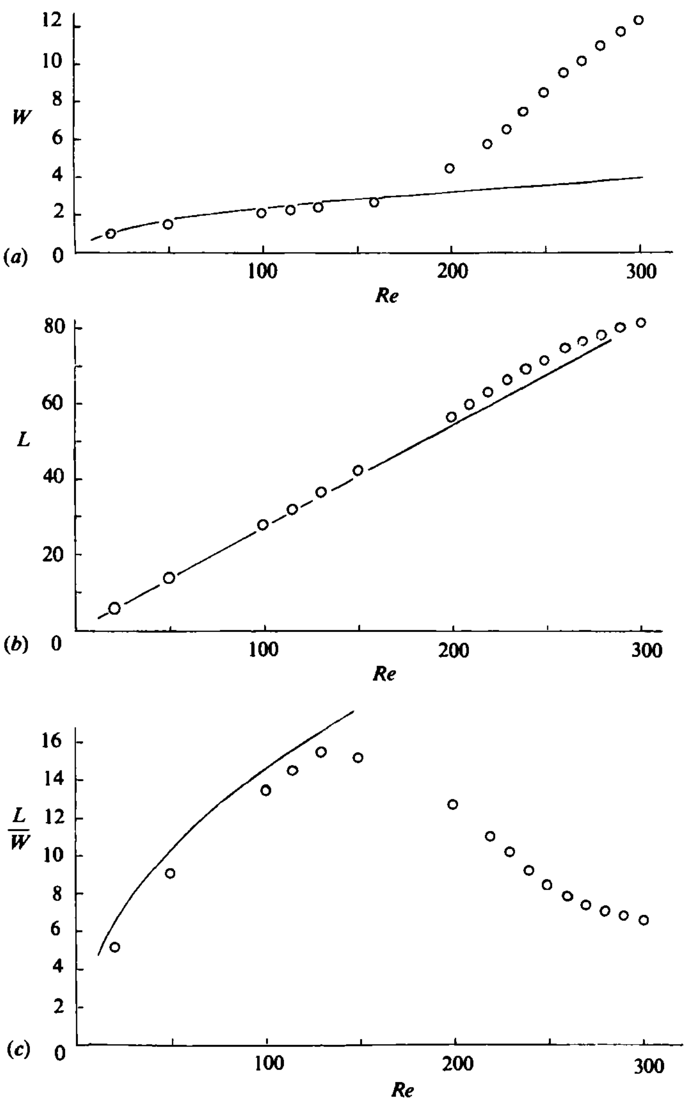

FIGURE 1. Variation of (*a*) eddy half-width W, (*b*) eddy length L and (*c*) the ratio L/W with Reynolds number. Circles are Fornberg's (1980) and (1985) results. The continuous lines are Smith's (1979) results: $W = 0.233\,Re^{\frac{1}{2}}$, $L = 0.34\,Re$ and $L/W = 1.46\,Re^{\frac{1}{2}}$.

flow which suggests that L/W may be closer to $O(1)$ than to $O(Re^{\frac{1}{2}})$ as in Smith's account. The description is in accord with Fornberg's results. The limit as $Re \rightarrow \infty$ is briefly discussed.

At the stage of final corrections to this paper, the author became aware of another proposal for the flow field by Smith (1985). Smith's new model is based on large strong eddies, bounded by vortex sheets, as described by Sandovskii (1971). There are differences from the model described in this paper, many of which are questions of interpretation and differences of opinion which are unlikely to be resolved until further analysis and computation are performed.

2. Vorticity in the flow

All vorticity in the eddy system and wake is fed by the shear layers separating from the cylinder's surface. We are not concerned here with details of separation and it appears that Smith's (1979) detailed account is appropriate. The shear-layer width grows with distance s like $Re^{-\frac{1}{2}}(as)^{\frac{1}{2}}$. If the flow near the rear of the cylinder is no more than that induced by the entrainment into the shear layer, then each shear layer follows an initially parabolic path, as described by Smith.

The shear layers entrain fluid on both sides. The velocity component normal to a layer at its surface is $O[U\,Re^{-\frac{1}{2}}(a/s)^{\frac{1}{2}}]$. At a distance s from the cylinder the accumulated entrained volume flux is thus $O[U\,Re^{-\frac{1}{2}}(as)^{\frac{1}{2}}]$. All the fluid entrained into the shear layers from the eddy must be returned to the eddy since the flow is steady. For parabolic growth of distance between the layers this implies a mean velocity in the eddy toward the cylinder of $O(U\,Re^{-\frac{1}{2}})$. Since entrainment is from both sides of the shear layer the streamline bounding the eddy is near the centre of the shear layer.

In a shear layer the vorticity has its maximum at the centre and is of order (velocity difference)/(width of shear layer). Thus the shear-layer vorticity is

$$O[U\,Re^{\frac{1}{2}}/(as)^{\frac{1}{2}}].$$

If the eddy has $o(U)$ velocities, then as the shear layer reaches the closure of the eddy the vorticity is

$$O[U\,Re^{\frac{1}{2}}/(aL)^{\frac{1}{2}}]. \tag{1}$$

This vorticity is being fed into the eddy, and once in it contributes to determining the eddy's shape.

An alternative possibility is that there is sufficient vorticity in the eddy that the fluid velocity just inside it approaches that of the free stream, and the shear layer is eliminated. In this case a typical vorticity in the eddy and on the bounding streamline is just

$$O(U/W). \tag{2}$$

There are three possibilities:

(i) the shear layer persists all the way to the eddy closure;

(ii) the shear layer is eliminated part way along the eddy boundary;

(iii) there is no shear layer and there are $O(1)$ velocities in the eddies. This is the case appropriate for Prandtl–Batchelor eddies of constant vorticity.

In all three cases the dividing streamline approaches the rear stagnation point in fluid of appreciable vorticity. As a first approximation it is reasonable to assume the vorticity distribution to be constant within an $O(a)$ distance of that streamline, since for the case (i) it is probably close to the maximum of the vorticity distribution across the layer; in case (ii) and (iii) there is likely to be little vorticity variation anyway because vorticity will have diffused into the fluid which travels on into the wake. Thus a solution is needed for a stagnation point between the two areas with vorticity of opposite sign, around the opposing streamlines.

Such a solution is given by the velocity field

$$(u,\,v) = (Ax+U(y),\; -Ay), \tag{3}$$

which satisfies the Navier–Stokes equations if the vorticity is given by

$$U'(y) = B\,\mathrm{erfc}\left[y\left(\frac{A}{2\nu}\right)^{\frac{1}{2}}\right]+C, \tag{4}$$

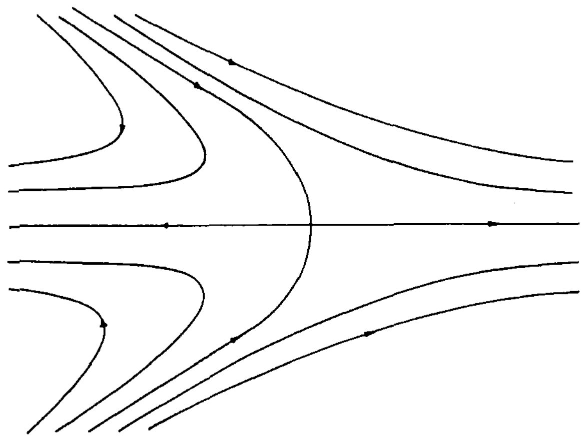

FIGURE 2. Sketch of the streamlines for the flow given by equations (3) and (5). The asymptotes of the dividing streamlines are $y = \pm(2A/\Omega)x$.

where B and C are constants determined by the uniform values of the vorticity as $y \rightarrow \pm\infty$. For example, in this case with vorticity $\mp\Omega$, for $y \rightarrow \pm\infty$,

$$U'(y) = -\Omega + \Omega \operatorname{erfc}\left[y\left(\frac{A}{2\nu}\right)^{\frac{1}{2}}\right]. \qquad (5)$$

A sketch of the streamlines is given in figure 2. The solution is one of a family of flows given by Jeffrey (1915).

This flow is a simple example of convection of vorticity balanced by viscous diffusion. Vorticity of opposite signs is convected towards $y = 0$. The only viscous effects are in the diffusive layer of constant width $\delta = O[(\nu/A)^{\frac{1}{2}}]$, centred on $y = 0$. Note also that despite the asymmetry of the streamlines about $x = 0$ the velocity on $y = 0$ is symmetrical about the origin.

Just outside the diffusive layer the streamlines bounding the eddies form a finite angle if $A = O(\Omega)$ and an angle approaching zero, a cusp, if $A = o(\Omega)$. This angle, or cusp, clearly depends on the eddy scale flow.

Between the closing stagnation point of the eddy and points between the centres of the two eddies, flow near the centreline has a component towards the centreline. The solution (5) cannot be expected to hold for all that distance but it is reasonable to expect that the convection of vorticity towards the centreline helps to contain the width of the diffusive layer to the same order of magnitude. This diffusive layer can be envisaged as a thin layer of near-zero vorticity between areas of positive and negative vorticity.

Once fluid travelling near the centreline passes the centre of the eddies, both diffusion and convection act together to increase the width of the low-vorticity layer. For high-Reynolds-number flows in this situation diffusion becomes unimportant and the low vorticity resulting from diffusion in the rear half of the eddy is convected along the streamlines. But does the flow in the eddy have a high Reynolds number? A lower bound on the velocity, deduced from entrainment into the shear layers, is $U\,Re^{-1}$, so the eddy Reynolds number must be no less than $O(U\,Re^{-1}W/\nu) = O(W/a)$. Thus unless $W = O(a)$ the flow has a high Reynolds number.

Now, consider whether Smith's (1979) model is consistent. With an eddy length $O(a\,Re)$ and low velocities in the eddy, the vorticity in the shear layer approaching

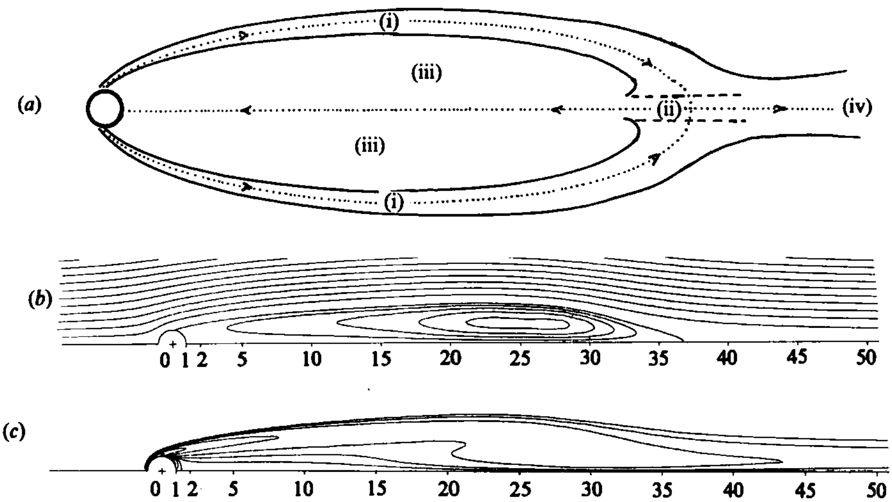

FIGURE 3. (*a*) A sketch of the flow field necessary for Smith's (1979) solution to be appropriate: (i) shear layers; (ii) diffusive layer; (iii) low-velocity eddy; (iv) wake. Fornberg's (1980) solution for $Re = 130$ ($Re = 260$ in his notation): (*b*) streamlines, Fornberg, figure 10(i) (with permission); (*c*) vorticity, Fornberg, figure 11(i) (with permission).

the rear stagnation point is, from (1), $O(U/a)$. An $O(1)$ fraction of this is on the inside of the bounding streamline. With fluid velocities of $O(U\,Re^{-\frac{1}{2}})$, the velocity in most of the eddy must be $O(U\,Re^{-\frac{1}{2}}/W) = O(U/a\,Re)$. The only way that this low level of interior vorticity can arise is if the $O(U/a)$ vorticity arriving from the shear layer is largely cancelled in the diffusive layer, that is, a flow like that sketched in figure 3(*a*). Such wholesale destruction of vorticity by viscosity does not seem to be an appropriate element of a high-Reynolds-number flow. Figures 3(*b*) and (*c*) show the Fornberg's (1980) solution for $Re = 130$, a flow which appears to accord with Smith's solution. There is some similarity with figures 3(*a*), and it is clear that diffusion of vorticity is important at this Reynolds number.

On the other hand for the type of flow indicated by (2), and (ii) or (iii) above, with stronger vorticity in the eddy, the solution (5) could well be appropriate for most of the rear part of the eddy since the vorticity approaching the diffusive layer would have nearly constant magnitude. Once fluid passes the centre of the eddy, viscosity is unimportant at the high Reynolds number within the eddy. All streamlines which pass through the diffusive layer carry little or no vorticity; thus a portion of the eddy near the rear of the cylinder will be nearly irrotational. This nearly irrotational region in the eddy near the cylinder need have no greater flow than that needed for entrainment in the shear layer, as assumed by Smith. The eddy may influence the position and strength of the separating shear layers, but would induce no more significant flow at the rear of the cylinder.

This view is consistent only if flow through the diffusive layer equals the flow entrained into the shear layers. The velocity near the end of the diffusive layer is $O(U)$ and hence $A = O(U/L)$ and the total flow is

$$O(U\delta) = O\left[U\left(\frac{\nu}{A}\right)^{\frac{1}{2}}\right] = O[(U\nu L)^{\frac{1}{2}}].$$

This feeds a length $O(L)$ of shear layer, supporting case (ii) since case (iii) becomes untenable unless $L = O(a)$, which is unlikely. Hence, since $L = O(a\,Re)$ by the

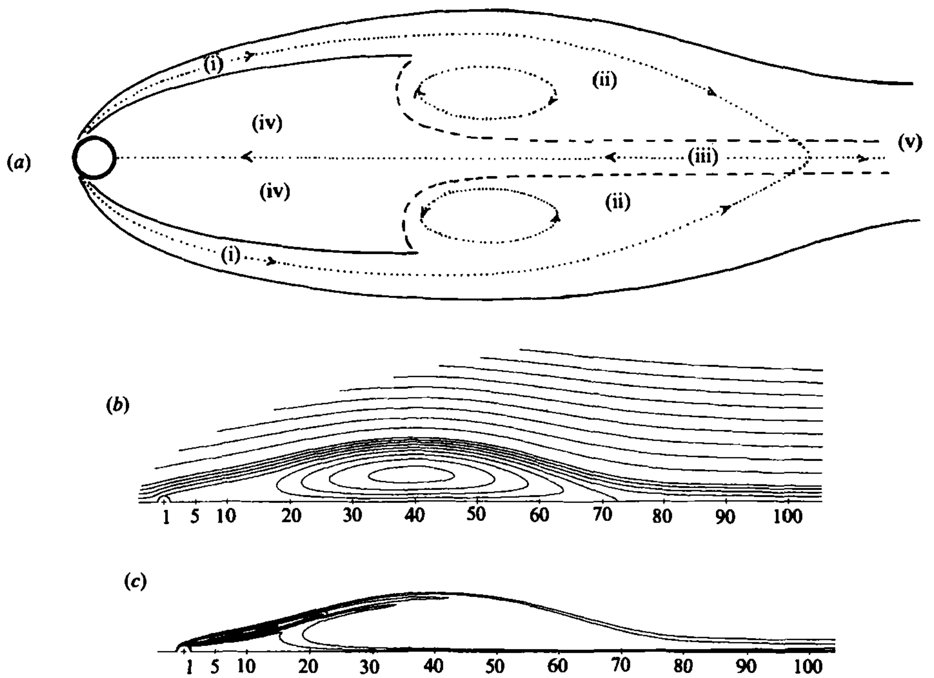

FIGURE 4. (*a*) A sketch of the flow field for eddies with appreciable vorticity: (i) shear layers; (ii) main region of vorticity; (iii) diffusive layer; (iv) low-vorticity region; (v) wake. Fornberg's (1985) solution for $Re = 250$ ($Re = 500$ in his notation): (*b*) streamlines, (with permission); (*c*) vorticity (with permission).

momentum arguments outlined in Smith (1979), the above discussion suggests the following:

$$L/W = O(1), \quad W = O(a\,Re), \quad \Omega = O(U/a\,Re), \quad A = O(U/a\,Re), \quad \delta = O(a). \quad (6)$$

A sketch of the flow that this discussion has led to is shown in figure 4(*a*). Fornberg's (1985) solution for $Re = 250$ is shown in figures 4(*b*) and (*c*). There is some resemblance:

(*a*) the shear-layer vorticity merges with the eddy vorticity midway along the eddy;

(*b*) there is a thin diffusive layer along the symmetry plane between the two areas of vorticity of opposite sense. However, it appears that δ may be smaller than $O(a)$;

(*c*) vorticity is almost uniform in the region of closed streamlines, except on those streamlines passing through the diffusive layer;

(*d*) the shear layers separating from the cylinder bound a region of fluid with low velocities and low vorticity.

Much of Smith's (1979) argument for wake eddies in the form of a slender ellipse depends on an assumption of small perturbations to the main flow field on the eddy lengthscale and hence small pressure variations within the eddy. The above considerations of vorticity and mass flow require no reference to pressure. The orders of magnitude (6) for the eddy indicate pressure differences across the eddy of $O(\rho U^2)$. Fornberg's (1985) figure 15 diagram of pressure fields shows Smith's description to be a reasonable approximation at $Re = 150$ but for higher Reynolds number $O(\rho U^2)$ pressure changes do occur.

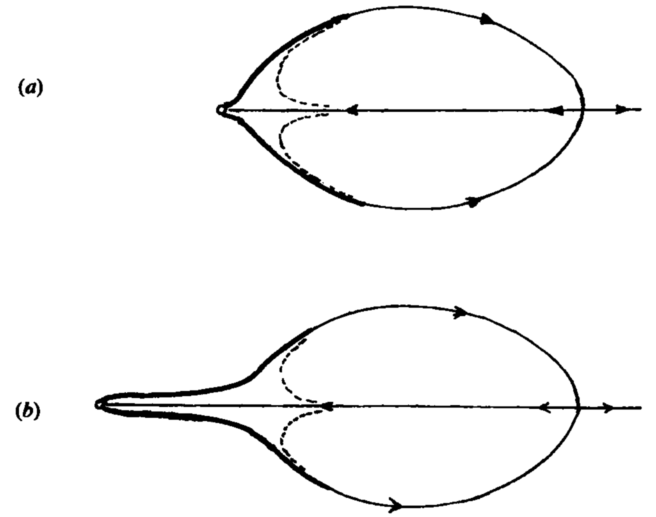

FIGURE 5. Possible inviscid limits for high-Reynolds-number flow. Sketches based on Saffman & Tanveer (1982) figure 1. (*a*) Cylinder $o(a\,Re)$ in front of main eddy. (*b*) Cylinder $O(a\,Re)$ in front of main eddy.

Some further support for a different, stronger flow in the eddies at high Reynolds number comes from calculations of the unsteady flow about a cylinder impulsively started from rest. Results from Son & Hanratty (1969) and Patel (1976) show that at the highest Reynolds numbers calculated 250 and 300 respectively, the growing eddy is more vigorous, wider and shorter than at $Re = 100$, for any given time after the start.

If the flow has a solution with shear layers like figure 3(*a*), figure 4(*a*) or any intermediate configuration it is hardly surprising that unsteady solutions prevail in a real fluid where shear layers are unstable.

3. The vorticity-induced translation velocity

This section is directed to further understanding the flow as $Re \to \infty$. The eddies grow with Re; consider them at very large Re. The cylinder is insignificantly small; the flow is dominated by the eddies and thus by the vorticity they contain. Nonetheless the eddies are moving at velocity U relative to distant fluid. The distribution of vorticity dominates the flow so its self-induced velocity of translation, V say, should equal U.

Now,

$$V = O(\Gamma/D), \tag{7}$$

where Γ is the circulation of a vortex (eddy) and is $O(\Omega S)$. In these expressions Ω is the magnitude of the vorticity, S is the area of vorticity of one sign and D is the distance between vortex centres. Insertion of the following orders of magnitude:

$$\Omega = O(U/W), \quad S = O(LW), \quad D = O(W),$$

which are appropriate for the model of figure 4(*a*), into expression (7) gives

$$V = O(UL/W). \tag{8}$$

With $V = U$ this contradicts any assumption of slenderness for the eddies, supporting the result at the end of the previous section.

The cylinder is at the front of the large eddies, so that the vortex-induced irrotational flow field in front of the cylinder leads to an incident velocity less than the free-stream velocity, and hence leads to a correction to the drag coefficient and all other quantities related to the flow near the cylinder. For example, with shear layers of length $O(L)$, figures 5(*a*) and (*b*) give two alternative sketches of the $Re \rightarrow \infty$ limit. The eddy shape is that of two touching areas of uniform vorticity of opposite sense taken from Saffman & Tanveer's (1982) correction to Pierrehumbert's (1980) solution. In figure 5(*a*) the distance of the sphere is $O(a\,Re)$ from the front of the eddy and in 5(*b*) it is $o(a\,Re)$ in front. The heavy lines indicate shear layers of length $O(a\,Re) = O(L)$. For figure 5(*a*) the drag coefficient would fall to zero as $Re \rightarrow \infty$ whereas for figure 5(*b*) it would fall to a finite value depending on the actual distance of the cylinder from the eddy. Fornberg (1985) finds $C_D = 0.430$ for $Re = 300$; this is below the value 0.5 of the Kirchhoff solution.

4. Conclusion

Discussions of vorticity and mass flow in the eddies behind a circular cylinder at high Reynolds number lead to a description of the flow consistent with Fornberg's (1985) numerical computations. Smith's (1979) theoretical description is satisfactory for $Re = 160$ but for larger Reynolds numbers the eddies become significantly wider and stronger.

The effect of large strong eddies as $Re \rightarrow \infty$ leads to an $O(1)$ change in the drag coefficient unless the cylinder is at a distance greater than $O(a\,Re)$ ahead of the eddies.

I am grateful to Dr. B. Fornberg for sending me a preprint of Fornberg (1985) and permitting use of his diagrams, and thank Professor F. T. Smith for a preprint of Smith (1985).

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